Elements of Calculus
and Analytic Geometry

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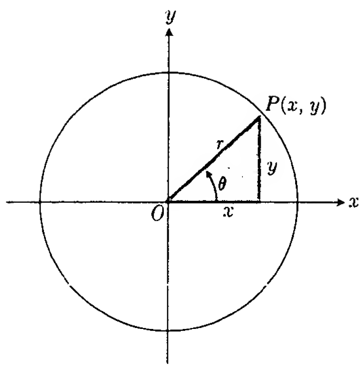
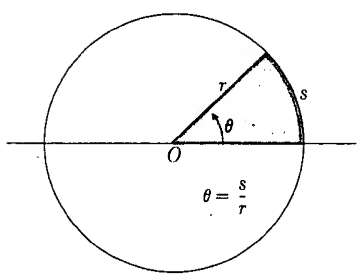


6.3 Angle θ in standard position.



6.4 In radian measure, $\theta = s/r$.

Similarly, for a right angle, $\theta = \pi/2$, we have $x = 0$, $y = r$; hence

$$\sin\frac{\pi}{2} = 1, \qquad \cos\frac{\pi}{2} = 0.$$

Radian measure

In all work with the trigonometric functions in the calculus, it is desirable to measure the angle in *radians*. The number of radians θ in the angle in Fig. 6.4 is defined as the number of "radius units" contained in the arc s subtended by the central angle θ; that is

$$\theta \text{ (in radians)} = \frac{s}{r}. \tag{4a}$$

This also implies that

$$s = r\theta \qquad (\theta \text{ in radians}). \tag{4b}$$

Another useful interpretation of radian measure is easy to get if we take $r = 1$ in (4b). Then the central angle θ, in radians, is just equal to the arc s subtended by θ. We may imagine the circumference of the circle marked off with a scale from which we may read θ. We think of a number scale, like the y-axis shifted one unit to the right, as having been wrapped around the circle. The unit on this number scale is the same as the unit radius. We put the zero of the scale at the place where the initial ray crosses the circle, and then we wrap the positive end of the scale around the circle in the counterclockwise direction, and wrap the negative end around in the opposite direction (see Fig. 6.5). Then θ can be read from this curved s-"axis."

Two points on the s-axis that are exactly 2π units apart will map onto the same point on the unit circle when the wrapping is carried out. For example, if $P_1(x_1, y_1)$ is the point to which an arc of length s_1 reaches, then arcs of length $s_1 + 2\pi$, $s_1 + 4\pi$, and so on, will reach exactly the same point after going completely around the circle one, or two, or more, times. Similarly P_1 will be the image of points on the negative s-axis at $s_1 - 2\pi$, $s_1 - 4\pi$, and so on. Thus, from the wrapped s-axis, we could read

$$\theta_1 = s_1,$$

or

$$\theta_1 + 2\pi,\ \theta_1 + 4\pi,\ \ldots,\ \theta_1 - 2\pi,\ \theta_1 - 4\pi,\ \ldots.$$

A unit of arc length $s = 1$ radius subtends a central angle of 57°18′ (approximately); so

$$1 \text{ radian} \approx 57°18'. \tag{5}$$

We find this, and other relations between degree measure and radian measure, by using the fact that the full circumference has arc length $s = 2\pi r$ and central angle 360°. Therefore

$$360° = 2\pi \text{ radians}, \tag{6a}$$

$$180° = \pi = 3.14159\ldots \text{ radians}, \tag{6b}$$

$$\left(\frac{360}{2\pi}\right)^{\circ} = 1 \text{ radian} \approx 57°17'44.8'', \tag{6c}$$

$$1° = \frac{2\pi}{360} = \frac{\pi}{180} \approx 0.01745 \text{ radian}. \tag{6d}$$